10/5/17

Άσκηση 3.10

$X_1, \dots, X_n \sim N(0,1)$ ανεξάρτητες

$\bar{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i$, και $\bar{X}_{n-k} = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i$

(i) $\frac{\bar{X}_k + \bar{X}_{n-k}}{2} \sim ?$

(ii) $k \cdot \bar{X}_k^2 + (n-k)\bar{X}_{n-k}^2 \sim ?$

(iii) $\frac{X_1^2}{X_2^2} \sim ?$

Λύση

(i) $\bar{X}_k \sim N\left(0, \frac{1}{k}\right)$ κ' $\bar{X}_{n-k} \sim N\left(0, \frac{1}{n-k}\right)$

$\frac{\bar{X}_k}{\sqrt{1/k}} \sim N(0,1)$ κ' $\frac{\bar{X}_{n-k}}{\sqrt{1/n-k}} \sim N(0,1)$

$\frac{1}{2}(\bar{X}_k + \bar{X}_{n-k}) \sim N\left(0, \frac{1}{4}(Var\bar{X}_k + Var\bar{X}_{n-k})\right)$

$W = \sum a_i X_i$

$Var(W) = \sum a_i^2 Var(X_i)$

$= N\left(0, \frac{1}{4}\left(\frac{1}{k} + \frac{1}{n-k}\right)\right)$

Συνεπώς ...

$$\left.\begin{aligned}
&\bar{X}_k \sim N\left(0,\tfrac{1}{k}\right) \sim \frac{\bar{X}_k}{\sqrt{1/k}} \sim N(0,1) \sim \sqrt{k}\,\bar{X}_k \sim N(0,1) \\
&\bar{X}_{n-k} \sim N\left(0,\tfrac{1}{n-k}\right) \sim \frac{\bar{X}_{n-k}}{\sqrt{1/n-k}} \sim N(0,1) \sim \sqrt{n-k}\,\bar{X}_{n-k} \sim N(0,1)
\end{aligned}\right\} (*)$$

ii) Από (*):

$$(\sqrt{k}\,\bar{X}_1)^2 + (\sqrt{n-k}\,\bar{X}_{n-k})^2 = k\bar{X}_k^2 + (n-k)\bar{X}_{n-k}^2 \sim \chi_2^2$$

$$\text{iii)}\quad \left.\begin{aligned}
&x_1 \sim N(0,1) \rightsquigarrow x_1^2 \sim \chi_1^2 \\
&x_2 \sim N(0,1) \rightsquigarrow x_2^2 \sim \chi_1^2
\end{aligned}\right\} \rightsquigarrow \frac{x_1^2/1}{x_2^2/1} = \frac{\chi_1^2/1}{\chi_1^2/1} \sim F_{1,1}$$

Άσκηση 1 (ασκήσεις από επανάληψη)

Τ.Σ. $X_1, \dots, X_{10}$ από $N(0,1)$ κ' $\bar{X}=\sum_{1}^{10} X_i/10$

και $X_{11}\sim N(0,1)$ και $Y_1=\sum_{i=1}^{10}(X_i-\bar{X})^2+X_{11}^2$

$Y_2=\sqrt{10}\,X_{11}\Big/\sqrt{\sum_{i=1}^{10}X_i^2}$ και

$Y_3=9(10\bar{X}^2+X_{11}^2)\Big/2\sum_{i=1}^{10}(X_i-\bar{X})^2$

(i) $P(Y_1\leq c_1)=0.01$

(ii) $P(Y_2\geq -c_2)=0.99$

(iii) $P(Y_3\geq c_3)=0.95$

Λύση

Επειδή ο πληθυσμός κανονικός

$$\frac{(n-1)S^2}{\sigma^2}\sim\chi^2_{n-1} \to \frac{(10-1)\frac{1}{10-1}\cdot\sum_{i=1}^{10}(X_i-\bar{X})^2}{1}\sim\chi^2_{10-1}$$

$\sum^{10} N^2(0,1)+N^2(0,1)$

$$\to \sum_{1}^{10}(X_i-\bar{X})^2\sim\chi^2_9 \Rightarrow Y_1=\sum_{1}^{10}(X_i-\bar{X})^2+X_{11}^2\sim\chi^2_{10}$$

Δηλαδή $Y_1\sim\chi^2_{10}$

$X_{11}\sim N(0,1)$ κ' $\sum_{1}^{10}X_i^2\sim\chi^2_{10} \Rightarrow$ ...

$$Y_2=\frac{\sqrt{10}\,X_{11}}{\sqrt{\sum_{1}^{10}X_i^2}}\equiv\frac{X_{11}}{\sqrt{\sum_{1}^{10}X_i^2/10}}\equiv t_{10}$$

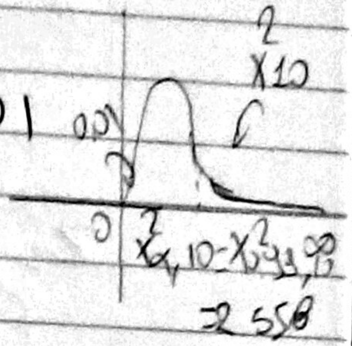
Ans. $Y_2 \sim t_{10}$

$\bar{X} \sim N\left(0, \frac{1}{10}\right) \to \frac{\bar{X}}{\sqrt{1/10}} \sim N(0,1) \to 10\bar{X}^2 \sim \chi^2_1$

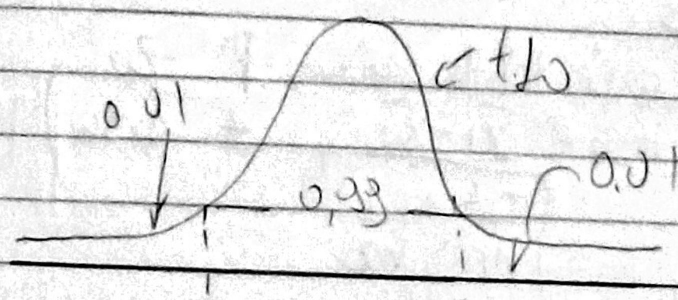
$10\bar{X}^2 + X_{11}^2 \sim \chi^2_2$ κ' $\sum_{1}^{10}(X_i-\bar{X})^2 \sim \chi^2_9 \to$

$$\to \frac{(10\bar{X}^2 + X_{11}^2)/2}{\sum_{i=1}^{10}(X_i-\bar{X})^2/9} = F_{2,9}$$

(i) $P(Y_1 \le c_1) = 0.01 \to P(\chi^2_{10} \le c_1) = 0.01$



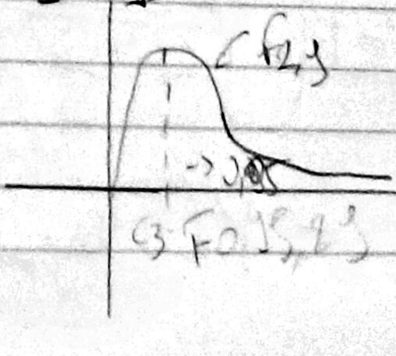
(ii) $P(Y_2 \ge -c_2) = 0.99 \to P(t_{10} \ge -c_2) = 0.99$



$c_2 = t_{\alpha,10} = t_{0.01,10} = 2.764$

(iii) $P(Y_3 \ge c_3) = 0.95 \to P(F_{2,9} \ge c_3) = 0.95$

$\to c_3 = F_{0.95,2,9} = \frac{1}{19.4} = 0.052$



Άσκηση 4.16
Δύο μηχανές M_1, M_2 : $n_1=100$, $n_2=100$, $\bar{x}_1=1.07$
$\bar{x}_2=1.18$, $\sigma_1=0.10$, $\sigma_2=0.12$

(i) Υπάρχει διαφορά ($\alpha=0.05$); " "

→ (ii) Ισχύς όταν μια μέση δ έχει μεγαλύτερο των μ_2 ($\mu_1-\mu_2=\delta$)

Λύση

$H_0: \mu_1-\mu_2=0$ v $H_a: \mu_1-\mu_2 \neq 0$

→ κριτ. $|z| \geq z_{\alpha/2}$ ($=z_{0.025}=1.96$)

$$\frac{\bar{x}_1-\bar{x}_2-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} \sim N(0,1) \text{ όταν } H_0 \text{ αληθ.}$$

$$z=\frac{1.07-1.18}{\sqrt{\frac{0.10^2}{100}+\frac{0.12^2}{100}}}=-7.042$$

σ' $7.042>1.96$
απορ. H_0, υπάρχει διαφορά

(ii) $\gamma=1-\beta$

$\beta=P(\text{δέχομαι } H_0 \mid H_0 \text{ αληθεύει})$

$$=P\left(-1.96 \leq \frac{\bar{x}_1-\bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} \leq 1.96 \,\middle|\, \mu_1-\mu_2=\delta\right)$$

z'

$$=P\left(-1.96-\frac{\delta}{\sqrt{\frac{0.10^2}{100}+\frac{0.12^2}{100}}} \leq \frac{\bar{x}_1-\bar{x}_2-\delta}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} \leq 1.96-\frac{\delta}{\sqrt{\frac{0.10^2}{100}+\frac{0.12^2}{100}}} \,\middle|\, z' \sim N(0,1)\right)$$

$$=P\left(\underbrace{-1.96-\frac{\delta}{0.0156}}_{\alpha_1} \leq z' \leq \underbrace{1.96-\frac{\delta}{0.0156}}_{\alpha_2} \,\middle|\, z' \sim N(0,1)\right)$$

$= .00 = \phi(02) - \phi(01)$